Detection methods with synthetic spectra

$\pi_* \mathbb{S} \longrightarrow \pi_* R$

§1 Low dimensional examples

§2 Periodification

§3 Height 2 situation

§4 Iterations & future ideas

(Notes will be made available!)

(j/w Christian Carrick)

§1 Low dimensional examples

Write $\eta : S^3 \longrightarrow S^2$ for the Hopf map with associated class $\eta \in \pi_1 \mathbb{S} \simeq \pi_1^{st}$ of the same name.

- There are many ways to show $\eta \neq 0 \in \pi_1 \mathbb{S}$.
- One can use Adem relations to show $\eta^2 \neq 0 \in \pi_2 \mathbb{S}$.

The next natural question is:

Does $\eta^3 \in \pi_3 \mathbb{S}$ vanish?

Perhaps you already know the answer...

(0) Compute $\pi_d \mathbb{S}$ for $d \in [0, 3]$.
(using A(N)SS, for example)

$\rightsquigarrow \eta^3 \neq 0.$

Great! But this tactic doesn't generalise very well.

Let's explore some methods using **ko**. ($\mathbb{S}$ → ko; ko = real topological k-theory)

$d \geq 0$, mod 8	0	1	2	3	4	5	6	7
$\pi_d(ko)$	$\mathbb{Z}$	$\mathbb{F}_2$	$\mathbb{F}_2$	0	$\mathbb{Z}$	0	0	0
generators	1	η	η^2	–	α	–	–	–
	β^k	$\eta\beta^k$	$\eta^2\beta^k$		$\alpha\beta^k$		…	$\forall k \geq 0$

w/ periodicity generator $\beta \in \pi_8(ko)$.

(Notice that this shows $\eta \neq 0 \neq \eta^2$, for example.)

However $\eta^3 = 0 \in \pi_3 ko = 0$, so it's not immediately clear that $\eta^3 \neq 0$ in $\pi_3 \mathbb{S}$.

To see this, we want to use the ANSS for ko.

I would like to consider $AN(ko)$ as a synthetic spectrum, but for most of this talk, it suffices to think of it as a filtered spectrum (this structure will be useful later)

Def^{n}: $AN(X) = \text{Déc}(X \otimes MU^{\otimes(\bullet+1)})$ ← Décalage; similar to SS of a filtered complex

in $\text{Fil}(Sp) := \text{Fun}(\mathbb{Z}^{op}, Sp) \xrightarrow{SS(-)} \text{SpecSeq}$.

In part., there is an exact functor

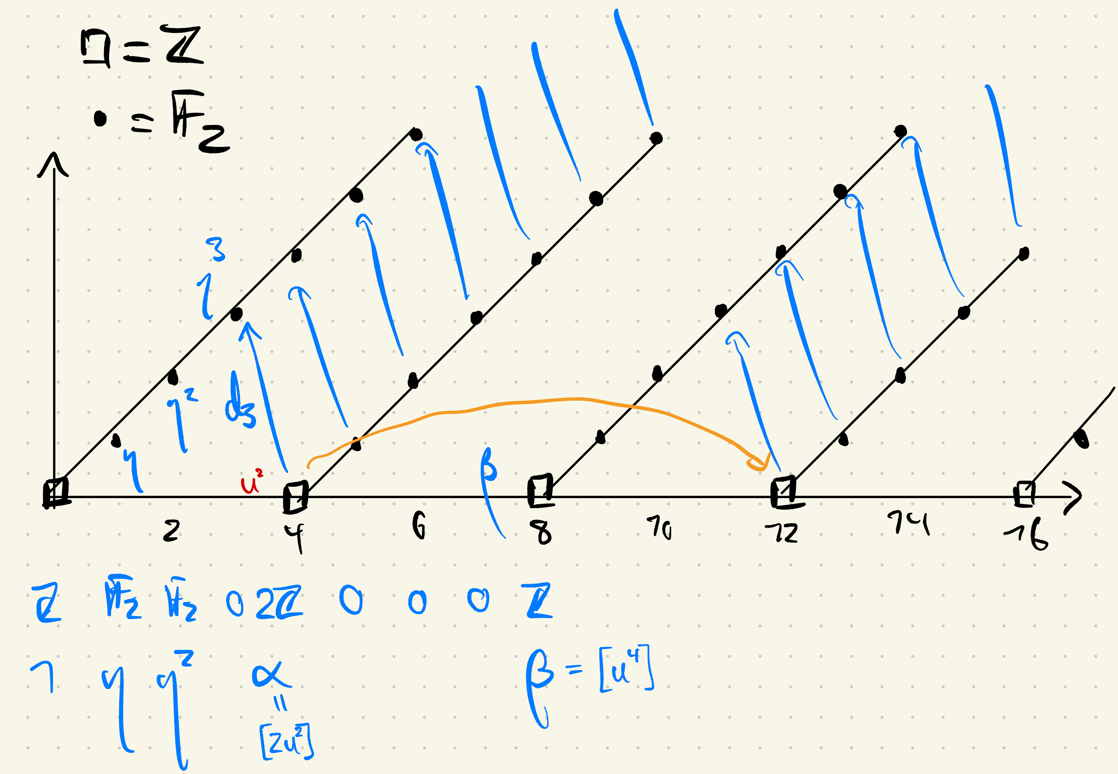
$$Gr^{*}(-) : \text{Fil}(Sp) \longrightarrow \prod_{\mathbb{Z}} Sp = Gr(Sp)$$

$$(\cdots \to X_{n+1} \to X_n \to \cdots) \mapsto (\ldots, X_n / X_{n+1}, \ldots)$$

with $\pi_s Gr^f(AN(X)) \simeq E_2^{s,f}\text{-}AN(X)$.

(See Sven van Nigtevecht's new paper for more!)
+ Hedenlund, Antieau

In particular, for $X = ko$, we have the familiar picture:



<u>Observation</u>: Although η^3 is zero in the image of $\pi_3 \mathbb{S} \longrightarrow \pi_3 ko$, η^3 is <u>nonzero</u> in the image of $E_2^{3,3}\text{-}AN(\mathbb{S}) \xrightarrow{\eta^3} E_2^{3,3}\text{-}AN(ko)$.

As η^3 is a permanent cycle in $AN(\mathbb{S})$, we say that η^3 lies in the

synthetic Hurewicz image of ko.

There are atleast 3 ways to argue now that $\eta^3 \neq 0$ in $\pi_3 \mathbb{S}$ using this information:

(1) **Filtration of source**

The d_3 which kills η^3 in $AN(ko)$ has filtration zero. However, $E_2^{*,0} AN(\mathbb{S}) = 0$ for $* \neq 0$ as $\pi_* \mathbb{S}$ is finite w/ $\pi_* BP$ is torsion free. Hence this d_3 in $AN(ko)$ cannot come from $AN(\mathbb{S})$.

$\rightsquigarrow$ either η^3 in $AN(\mathbb{S})$ is killed by a longer differential, or it survives. For degree reasons, it has to survive, so $\eta^3 \neq 0 \in \pi_3 \mathbb{S}$.

Moral: Differentials in $AN(R)$ hitting a class in the synthetic Hurewicz image of R with source in very low filtration, do not come from $\mathbb{S}$.

(2) Toda bracket argument

(come back to if there is time; basically assume $\eta^3 = 0$ in $\pi_3 \mathbb{S}$, and compute $\emptyset \neq \langle \eta^2, \eta, 2 \rangle \subseteq \pi_4 \mathbb{S}$ is nonzero in $\pi_4 ko$.)

(3) Deleting differentials

When completed at $p=2$, ko has the Adams operation ψ^3.

$\psi^3 : ko_2^\wedge \longrightarrow ko_2^\wedge$

A classical computation shows that

$\psi^3(\eta) = \eta$, $\psi^3(\alpha) = 3^2 \cdot \alpha$, $\psi^3(\beta) = 3^4 \beta$.

and these operations are multiplicative.

By functoriality, ψ^3 also acts on $AN(ko)$.

<u>Notice:</u> For the key $d_3(u^2) = \eta^3$

we have $\psi^3(\eta^3) = \eta^3$

and $\psi^3(u^2) = 3^2 u^2 \neq u^2 \in E_2\text{-}AN(ko)$.

In particular, if we define (in $Fil(Sp)$)

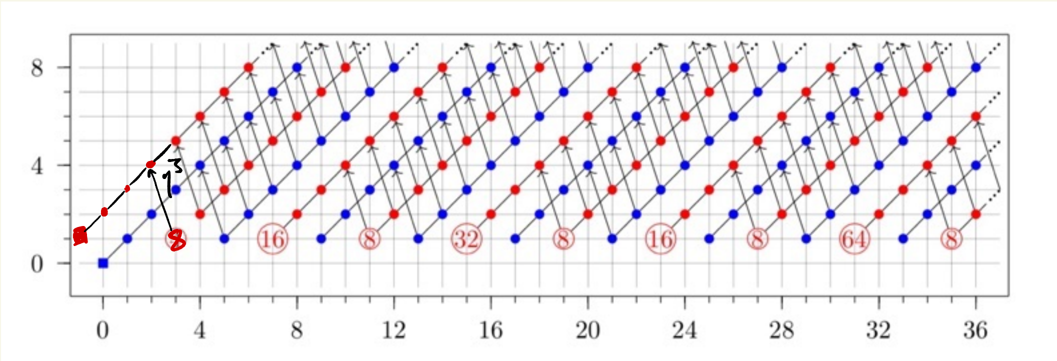
$$F = fib\left(AN(ko) \xrightarrow{\psi^3 - 1} AN(ko)\right),$$

$\eta^3 \mapsto \eta^3 - \eta^3 = 0$, $u^2 \mapsto 3^2u^2 - u^2 = 8u^2 \neq 0$

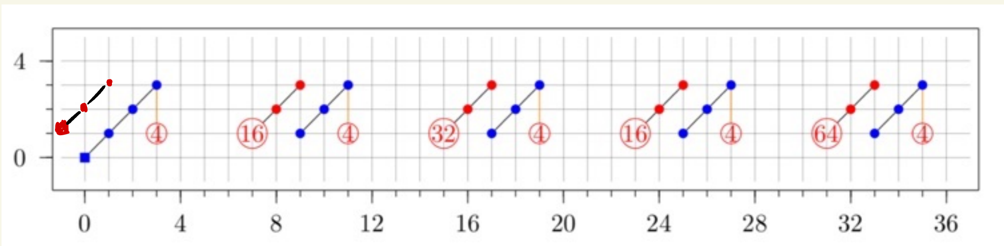
(a kind of "modified ANSS" for $fib(ko \xrightarrow{\psi^3 - 1} ko)$.)

then η^3 lifts to $E_2^{3,3}(F)$, but u^2 does <u>not</u> lift to $E_2^{4,0}(F)$. In fact, we can easily compute $E_2^{**}(F)$:

$$\left(\Sigma^{-1,1} AN(ko) \xrightarrow{\partial}\right) \quad F \longrightarrow AN(ko) \xrightarrow{\psi^3 - 1} AN(ko)$$



↓ w/ E_∞-page:



In particular, $\eta^3 \neq 0$ in $\pi_3 \mathrm{fib}(ko \xrightarrow{\psi^3-1} ko)$, so it's nonzero in $\pi_3 \mathbb{S}$.

($\mathrm{fib}(ko \xrightarrow{\psi^3-1} ko) = j$)

<u>Remarks:</u>
- This SS was easy, but $AN(j)$ is not well-understood.
- Notice that (3) is stronger than (1) & (2); (3) tells us <u>where</u> some elements are nonzero, i.e., in F. We can now think about applying (1)–(3) to F....

§2 Periodic classes

The application we've seen so far, that $\eta^3 \neq 0$, is not <u>that</u> interesting. What's nice though, is that these arguments <u>periodicity</u>.

There is a v_1^4-self map on $\mathbb{S}/2$,

$$v_1^4 : \mathbb{S}^8/2 \longrightarrow \mathbb{S}/2$$

$$\begin{array}{ccc} \mathbb{S}^8 & & \\ \downarrow{\scriptstyle \cdot 2} & & \\ \mathbb{S}^8 & \xrightarrow{\tilde v_1^4} & \mathbb{S}/2 \\ \downarrow & \nearrow{\scriptstyle v_1^4} & \\ \mathbb{S}^8/2 & & \end{array}$$

$$\pi_8 \mathbb{S}/2 \xrightarrow{\partial} \pi_7 \mathbb{S} \xrightarrow{\cdot 2} \pi_7 \mathbb{S}$$

$$\tilde v_1^4 \qquad\qquad 8\sigma \longmapsto 0$$

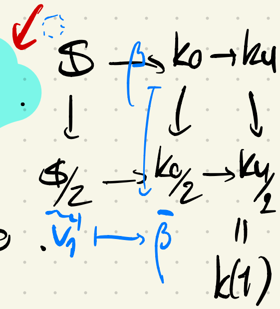
We can then define **Adams' μ-family**

$$\mu_{1+8d} : \mathbb{S}^{1+8d} \longrightarrow \mathbb{S}^{1+8d}/2 \xrightarrow{(v_1^4)^d} \mathbb{S}^1/2 \xrightarrow{\partial} \mathbb{S},$$

$\mu_{1+8d} \in \pi_{1+8d}\mathbb{S}$

Claim: $\mu_{1+8d} = \eta\beta^d \neq 0 \in \pi_{1+8d} ko$.

also $\eta\mu_{1+8d} = \eta^2\beta^d \neq 0 \in \pi_{2+8d} ko$.



↳ variations of arguments (1) – (3) above, show that $\eta^2\mu_{1+8d} \neq 0 \in \pi_{3+8d} \mathbb{S}$, (even though $\eta^2\mu_{1+8d} = \eta^3\beta^d = 0$ in ko)

In fact, writing $j = \mathrm{fib}(ko \xrightarrow{\psi^3 - 1} ko)$ we have:

(skipped to §3 in talk!)

Theorem [Adams, Toda]

The map $\pi_* \mathbb{S} \xrightarrow{(*)} \pi_* j$ is split surjective (in degrees ≥ 2).

Theorem [Carrick – D.]

The map $AN(\mathbb{S}) \longrightarrow mAN(j)$

$$mAN(j) = \mathrm{fib}\big(AN(ko) \xrightarrow{\psi^3 - 1} AN(ko)\big)$$

is split surjective, and induced a filtration on $\pi_* j$ s.t. ⊛ is a split surjection of filtered abelian groups.

In general, to detect v_n-periodic families in $\pi_* \mathbb{S}$ using these ideas we need:

- To know there exists v_n-periodic classes in $\pi_* \mathbb{S}$ to check for (we need these v_n^d-self maps).
- A v_n^d-periodic spectrum with a rich synthetic Hurewicz image, preferably with some operations as well.

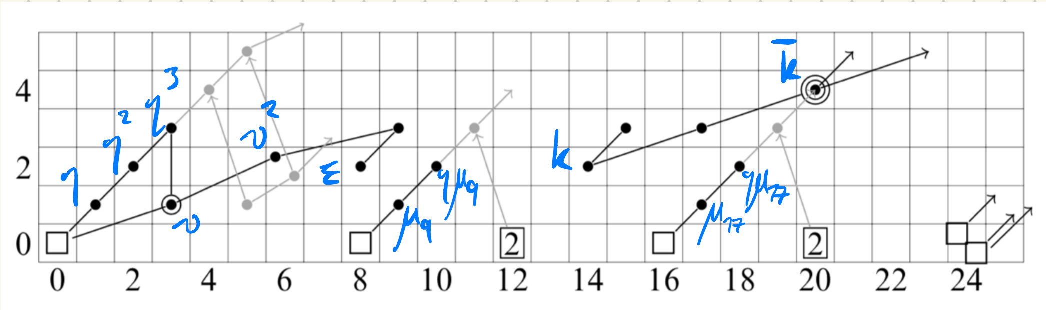
§3 Height 2 generalisations

At height 1 we had ko, at height 2 we have

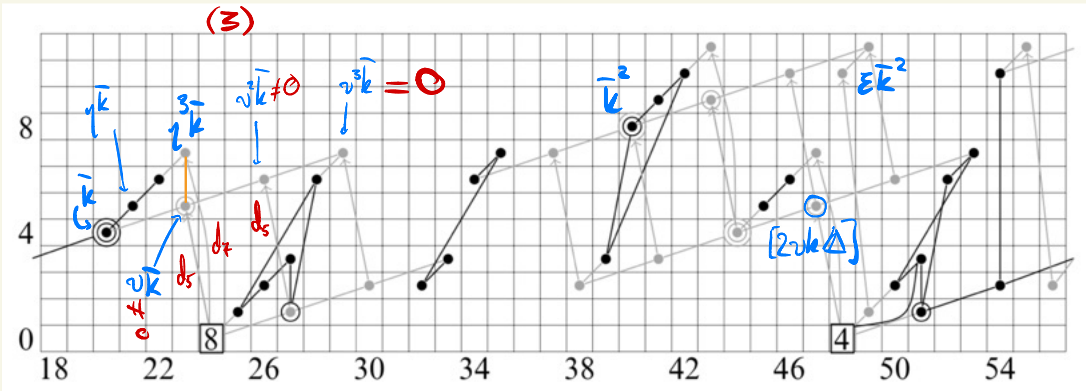
tmf = Hopkins' ring of topological modular forms.

Instead of being 8-periodic, tmf is 576-periodic (or 192-periodic when 2-completed).

ANSS(tmf) [0,20]



(1) shows $v^2\bar{k} \neq 0$ in $\pi_* S$



$\rightsquigarrow v\bar{k} \neq 0$ by (1) as this diff. has source in filt 0

also $v^2\bar{k} \neq 0$ by either (1), as we know the 1-line of the ANSS(S), or by (3) using

$$tmf \underset{w \circ can}{\overset{can}{\rightrightarrows}} tmf_0(3)$$

$\circlearrowleft w$ Atkin-Lehner invol.

$\nwarrow \Gamma_0(3)$-level structure

where $w \circlearrowright tmf_0(3)$ is the Atkin-Lehner involution on $tmf_0(3)$.

$\rightsquigarrow$ these nonvanishing results also periodicity using $144 = v_2^{32}$-self maps on $S/(3, v_1^4)$ of Behrens–Hill–Hopkins–Mahowald.

Altogether, Christian Carrick & I could show:

Theorem: The above techniques applied to $A\!N(tmf)$ produce 125 nonzero 192-periodic families in $\pi_*\mathbb{S}$ which are not detected in $\pi_* tmf$.

Cor: There are exotic d-spheres for all

$$d \equiv_{192} 72, 144, 168.$$

Pf [Outline]

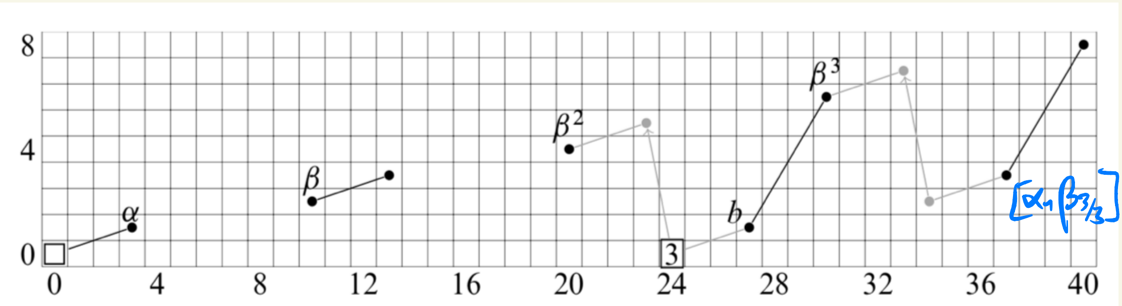
1) Compute some of synthetic Hurewicz image of tmf.

2) Run through versions of (1) - (3).

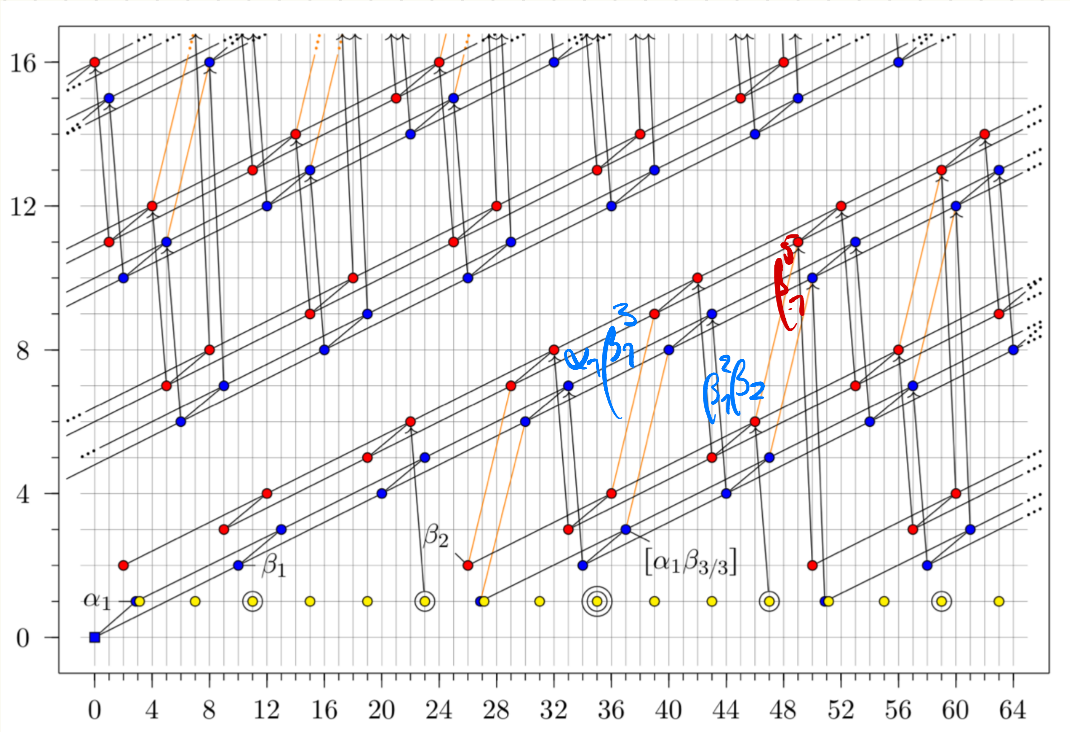
$\square$

§4 Further iterations

The above result was for $p=2$, but $\pi_* \mathrm{tmf}$ is also interesting at $p=3$.



Christian and I computed the Hurewicz image of $\mathrm{fib}\left(\mathrm{tmf}_3 \xrightarrow{\,t^2-1\,} \mathrm{tmf}_3\right)$, and then later realised that we could now iterate (1) – (2) above using this fibre:



In fact, I went further, and not just applied (1)–(2), but also periodified differentials.

Th [Shimomura, D.]

- $\prod_{i=1}^{M} \beta_{1+9s_i} \neq 0$ iff $M \leq 5$
- $\beta_{2+9s} \prod_{i=1}^{N} \beta_{1+9s_i} \neq 0$ iff $N \leq 2$.

\+ a whole bunch of similar results & questions.

Next steps:

The map $\psi^2 : TMF \longrightarrow TMF$ fits in a truncated cosimplicial diagram

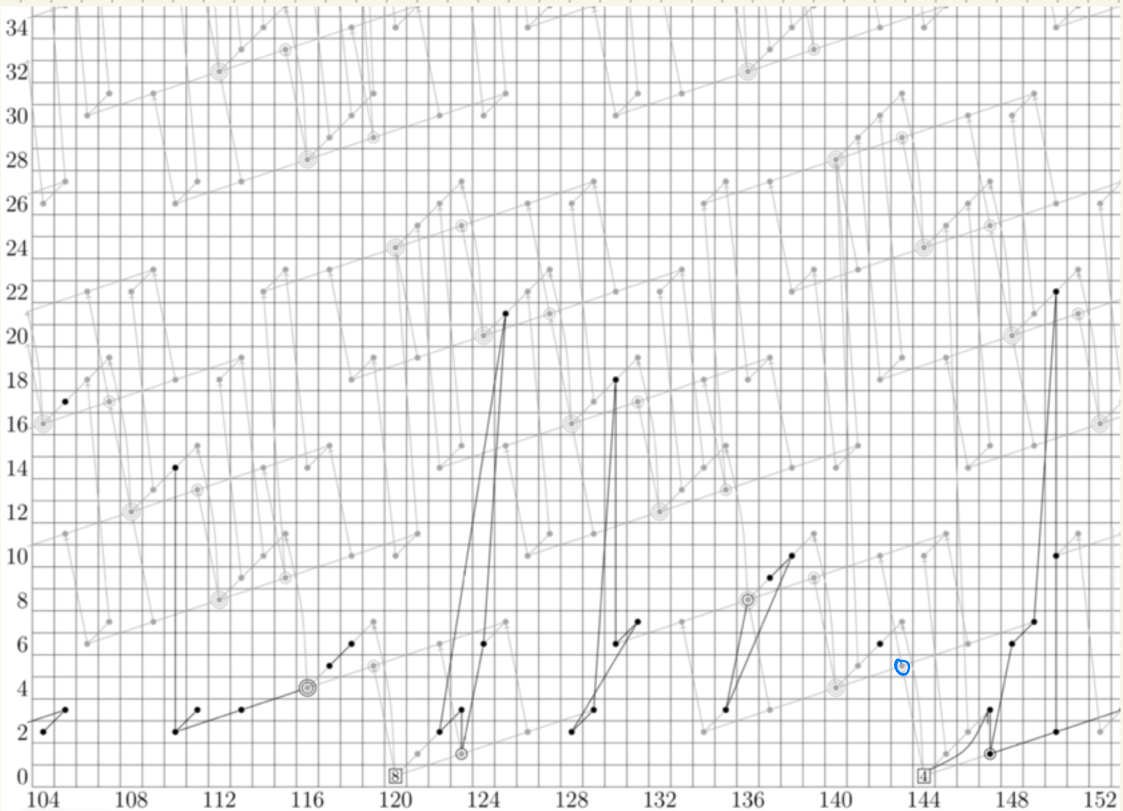
$$Q(2)^\bullet : \; TMF \; \underset{(\mathrm{id},\, \mathrm{con})}{\overset{(\psi^2,\, w\circ \mathrm{con})}{\rightrightarrows}} \; \begin{matrix} TMF \\ \times \\ TMF_0(2) \end{matrix} \; \begin{matrix} \xrightarrow{w} \\ \xrightarrow{\mathrm{con}} \\ \xrightarrow{\mathrm{id}} \end{matrix} \; TMF_0(2)$$

defined by Behrens.

Future:

- Use $Q(2) = \lim Q(2)^\bullet$ $(\sim Q(N))$ to better study $\pi_* \mathbb{S}$.
- Do this at $p=2$ and $p \geq 5$ too.
- Use similar tricks with E_n^{hH}.
- Continue this motivically, synthetically, equivariantly, ...

Thank you for listening! :)



$$0 \neq \nu^3 k = \eta^{\varepsilon k} \in \pi_{23}\mathbb{S}$$

$\nu^2 \leftarrow \{\nu^2_k\}$

$k \leftarrow \{k_n\}$

$\varepsilon \leftarrow \{\varepsilon_n\}$

$$\nu k \{\nu^2_k\} \neq 0$$

$$\overset{?}{=}$$

$$\nu^3 \{k_n\} \neq 0$$

$$\overset{?}{=}$$

$$\eta k \{\varepsilon_n\} \neq 0$$